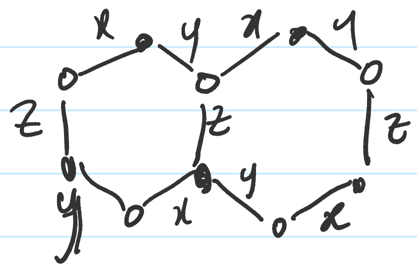
# Kitaev:

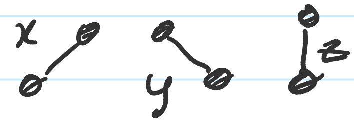
Monday, 4 April 2022 2:41 PM

**What is it?** Spin $\frac{1}{2}$ system,
spins $(\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle)$ are located
on vertices of a Hexagonal grid:




$$\mathcal{L} \cong (\mathbb{C}^2)^{\otimes n}$$

undirected links:




- Solid vs Hollow gives odd/even sublattices, useful for orientation

The Hamiltonian is given by associating energies to adjacent links with type depending on direction

$$H = -J_x \sum_{x\,\text{links}} \sigma_i^x \sigma_j^x - J_y \sum_{y\,\text{links}} \sigma_i^y \sigma_j^y - J_z \sum_{z\,\text{links}} \sigma_i^z \sigma_j^z$$

where $\sigma_i^\alpha$ represent $su(2)$, i.e. satisfy pauli matrix relations:

$$\sigma^x \sigma^y \sigma^z = i, \quad \{\sigma^\alpha, \sigma^\beta\} = 2\delta^{\alpha\beta}$$

$$\Rightarrow \quad \sigma^x \sigma^y = i\sigma^z \quad \sigma^y \sigma^z = i\sigma^x \quad \sigma^z \sigma^x = i\sigma^y$$

+ different sites commute,

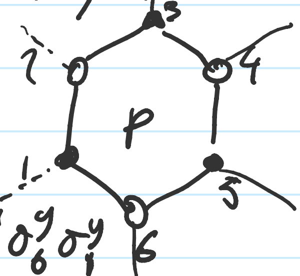
we can simplify notation by writing

$$\alpha_{ij} \begin{cases} x & ij \text{ have an } x \text{ link} \\ y & ij \text{ have a } y \text{ link} \\ z & ij \text{ have a } z \text{ link} \end{cases}$$

and writing $K_{ij} = \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}}$, and writing $K_{ij} = \sigma_i^{\alpha}\sigma_j^{\alpha}$,

gives $H = -\sum_{\substack{ij \\ \text{links}}} J_{\alpha_{ij}} K_{ij}$

Can express "flux" through a hexagon as



$$W_p = K_{12}K_{23}K_{34}K_{45}K_{56}K_{61}$$

$$= \sigma_1^z\sigma_2^z\sigma_2^x\sigma_3^x\sigma_3^y\sigma_4^y\sigma_4^z\sigma_5^z\sigma_5^x\sigma_6^x\sigma_6^y\sigma_1^y$$

$$= \sigma_1^z(i\sigma_2^y)(i\sigma_3^z)(i\sigma_4^x)(i\sigma_5^y)(i\sigma_6^z)\sigma_1^y$$

$$= i^5\sigma_1^z\sigma_1^y\sigma_2^y\cdots\sigma$$

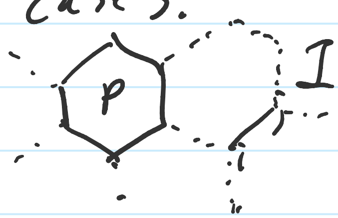
$$= i^5(-i)\sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z$$

$$= \sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z$$

– this is the "outward" spin at each site.

Note that: $[W_p, K_{ij}] = 0$:

3 cases:



I isolation: $\sigma_i\sigma_j$ at distinct sites to spins in $W_p$.

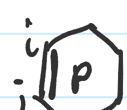
II Normality

wlog, $K_{ij} = K_{i1}$, $W_p = \sigma_1^x \cdots \sigma_6^z$

$$[K_{i1}, W_p] = \sigma_i^x\sigma_1^x\sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z - \sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z\sigma_1^x\sigma_i^x$$

$$= (\sigma_i^x - \sigma_i^x)\,\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z$$

III Tangency:



wlog $K_{ij} = K_{12} = \sigma_1^z\sigma_2^z$.

$$[K_{12}, W_p] = \sigma_1^z\sigma_2^z\sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z$$

$$- \sigma_1^x\sigma_2^y\sigma_3^z\sigma_4^x\sigma_5^y\sigma_6^z\sigma_1^z\sigma_2^z$$

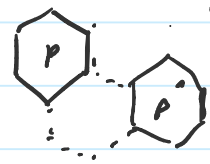
$$[H_{12}, W_p] = \sigma_1^x \sigma_2^y \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z - \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \sigma_1^z \sigma_2^z$$

$$= (\sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z)(\sigma_1^z \sigma_1^x \sigma_2^z \sigma_2^y - \sigma_1^x \sigma_1^z \sigma_2^y \sigma_2^z)$$

$$= \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z [(i\sigma_1^y)(-i\sigma_2^x) - (-i\sigma_1^y)(i\sigma_2^x)]$$

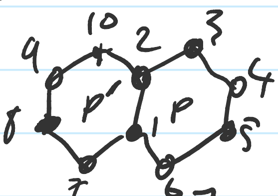
$$= 0$$

Additionally, if $p \neq p'$, $[W_p, W_{p'}] = 0$ : 2 cases:




I: Distant

no matching sites, all spins commute.




II: Adjacent wlog

matching sites are $i=1, j=2$: $[W_p, W_{p'}]$

$$= \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \sigma_8^x \sigma_9^y \sigma_{10}^z \sigma_2^x \sigma_1^y \sigma_7^z - \sigma_8^x \sigma_9^y \sigma_{10}^z \sigma_2^x \sigma_1^y \sigma_7^z \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z$$

$$= (\sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \sigma_7^z \sigma_8^x \sigma_9^y \sigma_{10}^z)(\sigma_1^x \sigma_1^y \sigma_2^y \sigma_2^x - \sigma_1^y \sigma_1^x \sigma_2^x \sigma_2^y)$$

$$= (\sigma_3^z \cdots \sigma_{10}^z)[(i\sigma_1^z)(-i\sigma_2^z) - (-i\sigma_1^z)(i\sigma_2^z)]$$

$$= 0$$

- $W_p$ are commuting, diagonalizable (given $\sigma_i^a$ are hermitian) operators, $\therefore$ $\mathcal{L}$ has a basis of simultaneous eigenvectors of $W_p$

As $W_p^2 = (\sigma_1^x)^2 (\sigma_2^y)^2 (\sigma_3^z)^2 (\sigma_4^x)^2 (\sigma_5^y)^2 (\sigma_6^z)^2$
$= 1$,

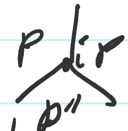
$= 1,$

these eigen values are $\pm 1$.

Therefore, $\mathcal{L} = \bigoplus_{w_1,\dots,w_m} \mathcal{L}_{w_1,\dots,w_m}$

($m = \#$Hexagons) where $w_1,\dots,w_m = \pm 1$, i.e., $w \in \{-1,1\}^m$ and $\mathcal{L}_{w_1,\dots,w_m}$ is the space where for each Hexagon $p_i$, $W_{p_i}$ acts as $w_i$.

there are $2^m$ terms in the sum above, and as




$6m \approx 3n$ (each hexagon has 6 vertices, and each vertice

we can approximate $\dim \mathcal{L}_{w_1,\dots,w_m} \approx \frac{2^n}{2^m} \approx 2^{\frac{n}{2}}$.

(we later show $\dim \mathcal{L}_{w_1,\dots,w_m}$ does not depend on $w_1,\dots,w_m$ choice),

So, this decomposition does not solve the problem yet. To proceede we express the degrees of freedom as real (majorana) fermions.

In this form the Hamiltonian can be written as a quadratic form, and an exact solution can be found.

## Spin - Fermion transformation:

A fermionic system with $n$ modes is described by anhilation and creation operators, $a_k$, $a_k^\dagger$, for $k = 1,\dots,n$.

$$\{a_k, a_\ell\} = \{a_k^\dagger, a_\ell^\dagger\} = 0, \quad \{a_k, a_\ell^\dagger\} = \delta_{k\ell}$$

$$\{a_k, a_l\} = \{a_k^\dagger, a_l^\dagger\} = 0, \quad \{a_k, a_l^\dagger\} = \delta_{kl}$$

We can also rewrite this system in terms of "majorana" operators:

$$c_{2k} = a_k + a_k^\dagger \quad , \quad c_{2k-1} = \frac{1}{i}(a_k - a_k^\dagger) \quad - \text{ hermitian}$$

$$\{c_k, c_l\} = 2\delta_{kl} : \quad \text{pf}$$

$k \neq l$:
$$\{c_{2k}, c_{2l}\} = \{a_k + a_k^\dagger, a_l + a_l^\dagger\} = 0$$
$$\{c_{2k}, c_{2l-1}\} = -i\{a_k + a_k^\dagger, a_l - a_l^\dagger\} = 0$$
$$\{c_{2k-1}, c_{2l-1}\} = -\{a_k - a_k^\dagger, a_l - a_l^\dagger\} = 0$$

$$\begin{aligned}
\{c_{2k}, c_{2k}\} &= \{a_k + a_k^\dagger, a_k + a_k^\dagger\} \\
&= 2(a_k^2 + (a_k^\dagger)^2 + \{a_k, a_k^\dagger\}) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\{c_{2k-1}, c_{2k-1}\} &= -\{a_k - a_k^\dagger, a_k - a_k^\dagger\} \\
&= -2(a_k^2 - a_k a_k^\dagger - a_k^\dagger a_k + a_k^{\dagger 2}) \\
&= 2\{a_k, a_k^\dagger\} = 2
\end{aligned}$$

$$\begin{aligned}
\{c_{2k}, c_{2k-1}\} &= -i\{a_k + a_k^\dagger, a_k - a_k^\dagger\} \\
&= -i[a_k^2 - a_k a_k^\dagger + a_k^\dagger a_k + (a_k^\dagger)^2 \\
&\qquad + a_k^2 + a_k a_k^\dagger - a_k^\dagger a_k + (a_k^\dagger)^2] \\
&= 0
\end{aligned}$$

(This means that $c_i$, $i = 1, \dots, 2n$ generate a $(\pm 0, \mp 2n)$ signature clifford algebra), and have identical anticommutators.

At each site, consider 4 fermionic modes, labelled $b^x, b^y, b^z, c$, which act on $(\mathbb{C}^2)^{\otimes 2}$, a dim 4 fock space, $\tilde{M}$.

We define the operator $D = b^x b^y b^z c$,

$$D^2 = b^x b^y b^z c\, b^x b^y b^z c$$
$$= -b^x b^y b^z b^x b^y b^z c^2 = -b^x b^y b^x b^y b^{z2}$$
$$= (b^x)^2 (b^y)^2 = 1$$

$D$ has evals $\pm 1$ define $M$ as the 1-eigenspace of $D$, and as

$$D\phi = \phi \qquad D\, c\phi$$
$$= b^x b^y b^z \phi$$
$$= c\, c\, b^x b^y b^z \phi$$
$$= -c\, b^x b^y b^z c\phi$$
$$= -cD\phi = -c\phi$$

$\therefore$ $c$ interchanges $\pm 1$ $D$ eigenvalue, $c: M \to \tilde{M}$
gives $M^\perp \cong M$, $\dim M = 2$

$M$ is the physical subspace, $\tilde{M}$ is the extended space.

We can extend $SU(2)$ action on $M$ to a $(b^\alpha, c)$ action on $\tilde{M}$:

define $\tilde{\sigma}^\alpha = i b^\alpha c$ for $\alpha = x, y, z$.

then: $(\tilde{\sigma}^\alpha)^\dagger = -i c b^\alpha = i b^\alpha c = \tilde{\sigma}^\alpha$

$$(\tilde{\sigma}^\alpha)^2 = -c b^\alpha c b^\alpha = c^2 b^{\alpha 2} = 1$$

$$\tilde{\sigma}^x \tilde{\sigma}^y \tilde{\sigma}^z = (-icb^x)(-icb^y)(-icb^z)$$
$$= (-i)^3 (c^2 b^x b^y b^z c)$$
$$= iD$$

So: $\tilde{\sigma}^\alpha\big|_{\tilde{M}}$ represent $SU(2)$, as $D\big|_{\tilde{M}} = 1$

Allowing us to consider the spin system $\mathcal{L}_1 = \mathbb{C}\tilde{M}$ as a Majorana system.

For multi-spin systems we replace each spin with a $\tilde{M}$, obtaining $\tilde{\mathcal{L}} = (\tilde{M})^{\otimes n}$ (dim $2^{2n}$)

and we replace the Hamiltonian $H(\sigma_i^\alpha)$ with $\tilde{H}\{b_i^\alpha, c_i\} = H(\tilde{\sigma}_i^\alpha)$, $\tilde{\sigma}_i = i b_i^\alpha c_i$

The physical subspace is obtained by requiring $D_i = b_i^x b_i^y b_i^z c = 1$, i.e.

$$\mathcal{L} = \bigcap_{i=1}^{n} \ker(D_i - 1), \text{ and}$$

$$\tilde{H}(b_i^\alpha, c_i)\big|_{\mathcal{L}} = H(\sigma^\alpha)$$

Note that $[\tilde{H}, D_i] = 0$ as if $i \neq j$

$$[\tilde{\sigma}_j^\alpha, D_i] = i(b_j^\alpha c_j)(b_i^x b_i^y b_i^z c_i) - i(b_i^x b_i^y b_i^z c_i)(b_j^\alpha c_j) = 0,$$

and $[\tilde{\sigma}_i^\alpha, D_i] = i\, b_i^\alpha c_i b_i^x b_i^y b_i^z c_i - i\, b_i^x b_i^y b_i^z c_i b_i^\alpha c_i$
$= -i(b_i^\alpha b_i^x b_i^y b_i^z - b_i^x b_i^y b_i^z b_i^\alpha) c_i^2$

wlog $\alpha = x$. (permute $b_i^x b_i^y b_i^z$ to have $\alpha$ first) w.l.o.g., $\alpha = x$, (permute $b_i^x b_i^y b_i^z$ to have $\alpha$ first)

$$[\tilde{\sigma}_i^\alpha, D_j] = -i[b_i^x, b_i^x b_i^y b_i^z]$$
$$= -i (b_i^x)^2 b_i^y b_i^z - i\, b_i^x b_i^y b_i^z b_i^x$$
$$= -i\,(b_i^y b_i^z - b_i^y b_i^z)$$

Therefore all $[\tilde{\sigma}_j^\alpha, D_j] = 0$, so

$$[\tilde{H}(\tilde{\sigma}_j^\alpha), D_j] = 0$$

Fermionic Kitaev: $\hat{\sigma}_i^{\alpha_{ij}} \hat{\sigma}_j^{\alpha_{ij}} = \hat{K}_{ij}$

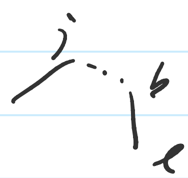
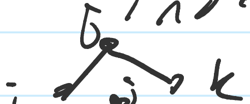
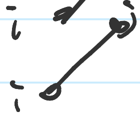
$$= -i(i\, b_j^\alpha b_k^\alpha) c_i c_k$$

$\hat{u}_{jk} = i\, b_i^{\alpha_{ij}} b_j^{\alpha_{ij}}$ lets us write

$$\hat{H} = \frac{i}{4} \sum_{j,k} \hat{A}_{jk} c_j c_k$$

where $\hat{A}_{jk} = \begin{cases} 2J_{\alpha_{jk}} \hat{u}_{jk} & j \text{ connected to } k \\ 0 & \text{else} \end{cases}$

note: $[\hat{u}_{ij}, \hat{u}_{kl} c_k c_l] = 0$:

- 3 cases: indep, isolated: 
adjacent:  , and
identical: 

TBA commutators

Therefore, $[\hat{u}_{ij}, \hat{H}] = 0$ and as $\hat{u}_{ij}$ is a self adjoint operator squaring to $\hat{u}_{ij}^2 = b_i^{\alpha} b_j^{\alpha} b_i^{\alpha} b_j^{\alpha} = -b_i^{\alpha} b_j^{\alpha} b_j^{\alpha} b_i^{\alpha} = 1$,

$\tilde{\mathcal{L}}$ splits into $\hat{u}_{ij}$ eigenspaces,

$$\tilde{\mathcal{L}} = \bigoplus_{u \in \{-1,1\}^{\text{\# edges}}} \tilde{\mathcal{L}}_u$$

$\tilde{\mathcal{L}}_u$ has $\hat{u}_{ij} \equiv u_{ij} = \pm 1$

on $\tilde{\mathcal{L}}_u$, $H = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k$

where $A_{jk} = \begin{cases} 2 J_{\alpha_{jk}} u_{jk} & j \leftrightarrow k \\ 0 & \text{else.} \end{cases}$

- this is exactly solvable, w/ groundstate $|\Psi_u\rangle$, but this state is non-physical.

$\hat{u}_{jk} D_j |\tilde{\Psi}_u\rangle = i b_j^{\alpha} b_k^{\alpha} \; b_j^x b_j^y b_j^z c_j |\tilde{\Psi}_u\rangle$

( wlog, $\alpha_{jk} = x$ ) $= - b_k^x b_j^y b_j^z b_j^z c_j$

$= - D_j \hat{u}_{jk} |\tilde{\Psi}_u\rangle$

$= - u_{jk} D_j |\tilde{\Psi}_u\rangle$

$\implies D_j$ - reverse $u_{jk}$!

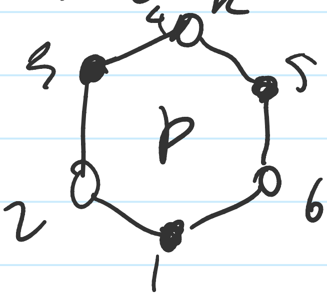
$= \ldots_{jk} D_j \ldots$ — reverse $u_{jk}$!

$$|\Psi_w\rangle = \prod_j \left(\frac{1+D_j}{2}\right)|\hat{\Psi}_u\rangle$$

characterised by choice of

$w_p = \pm 1$ defined as $w_p = \prod_{(j,k)\in\partial p} u_{jk}$

using convention of taking $u_{jk}$ w/ $j$ even, $k$ odd:



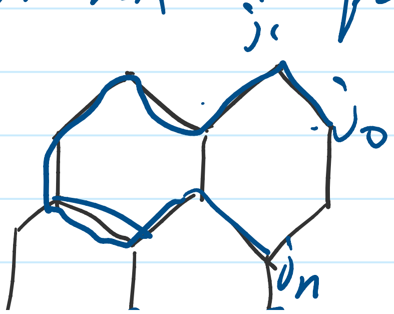

$$w_p = u_{21} u_{23} u_{43} u_{45} u_{65} u_{61}$$

- equivalent to $\tilde{W}_p = \prod_{\substack{(j,k)\in\partial p \\ j \text{ even}, k \text{ odd}}} \hat{u}_{jk}$ eigenvalues.

- $[\tilde{W}_p, D_j]$ & $[\hat{W}_p, \tilde{H}]$

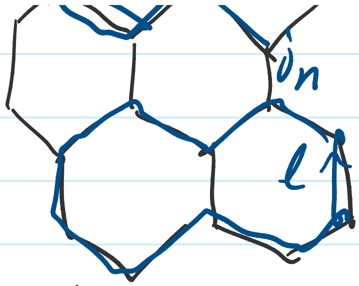
So $\tilde{W}_p$ physical eigenvalues give us decomposition of $\mathcal{L}$ & $\hat{W}_p|_{\mathcal{L}} = W_p$ so it restricts to

Wilson loops:




$$w(j_0, \ldots j_n) = K_{j_0 j_1} \ldots K_{j_{n-1} j_n}$$

corresponds to transfer of a fermion along path ...




corresponds to transfer of a fermion along path $i_0, \ldots, i_n$

For a closed oriented loop $\ell$, $W(\ell) = W(\ell_0, \ldots, \ell_n)$ is a wilson loop,
on honeycomb all loops have even length, e-vals $\pm 1$,
on lattices w/ odd loops, evals $\pm i$ instead.

this comes from time reversal,
$\{T, \sigma_i^a\} = 0$, $[T, b_i^a] = [T, c_i] = 0$.

leads to $W_\ell |\Psi\rangle = w_\ell |\Psi\rangle \Rightarrow$
$W_\ell T|\Psi\rangle = \bar{w}_\ell T|\Psi\rangle$

even: $w_\ell = w_\ell^*$

Note: wilson loops correspond to flux through enclosed area, $w_p = -1 \Leftrightarrow p$ carries a vortex.

T symmetry $\Rightarrow$ $\tilde{H}$ eigenspaces at least 2 fold degenerate.

Quadratic Hamiltonian on $\widetilde{\mathcal{L}}_u$

$$H(A) = \frac{i}{4} \sum_{j,k} A_{jk} c_j c_k \quad (\text{drop} \sim)$$

$A \in so(2m)$, & $\frac{i}{4}$-factor

chosen s.t. $\forall A, B \in \mathfrak{so}(2m)$,

$$[-iH(A), -iH(B)] = -iH(A, B)$$

i.e. $-iH : \mathfrak{so}(2m) \to \mathfrak{gl}(\tilde{\mathcal{L}}_u)$ is a $\mathfrak{so}(2m)$ rep, and operators $e^{-iH(A)}$ form $Spin(2m)$ (universal cover of $SO(2m)$,

$Spin(2m) = \{ a_{i_1} a_{i_2} \dots a_{i_{2n}} \in Cl(2m) \mid \|a_{i_r}\| = 1 \}$)

and $Spin(2m)$ acts on majoranas by conjugation, $Q = e^{A}$

$$e^{-iH(A)} c_k e^{-iH(A)} = \sum_j Q_{jk} c_j$$

factors through quotient $\{\pm 1\}$ to give a map / rep

$$SO(2m) \longrightarrow GL(2m)$$

given a $\mathbb{R}$-linear combination of Majorana operators, $f(x) = \sum_j x_j c_j$

(Treat $\underline{x}$ as a column vector). To find $|\Psi_\alpha\rangle$, need to write

$$H_{canonical} = \frac{i}{2} \sum_{k=1}^{m} \varepsilon_k b_k' b_k'' = \sum_{k=1}^{m} \varepsilon_k (a_k^\dagger a_k - \tfrac{1}{2})$$

$H_{canonical} = \frac{i}{2} \sum_{k=1} \epsilon_k b_k' b_k'' = \sum_{k=1} \epsilon_k (\ldots - \frac{1}{2})$

$b_k', b_k''$ "normal modes"

$$a_k = \frac{1}{2}(b_k' - i b_k''), \quad a_k^\dagger = \frac{1}{2}(b_k' + i b_k'')$$

$\therefore$ ground state given by $a_k |\psi_0\rangle = 0$

- obtained by $(b_1', b_1'', \ldots, b_m', b_m'')$

$$= (c_1, c_2, \ldots, c_{2m-1}, c_{2m}) Q$$

$Q \in O(2m)$, $Q$ solves

$$A = Q \begin{pmatrix} 0 & \epsilon_1 & & & \\ -\epsilon_1 & 0 & & & 0 \\ & & \ddots & & \\ & & & 0 & \epsilon_m \\ & & & -\epsilon_m & 0 \end{pmatrix} Q^T$$

- this exists because $A \in so(2m)$

so apply spectral thm turn conjugate pair imaginary e-vals into $2 \times 2$ $\begin{pmatrix} 0 & \epsilon_j \\ -\epsilon_j & 0 \end{pmatrix}$ blocks.

$\epsilon_k \longleftrightarrow iA$ spectrum

odd/even $Q$ cols $\longleftrightarrow$ Real/Im. pair of $A$ eigenvecs.

ground state energy equals

$$E = -\frac{1}{2} \sum_{k=1}^{m} \epsilon_k = -\frac{1}{4} \mathrm{Tr}\, |iA|$$

$|\ |$ defined w/ evals

ground state does not uniquely depend on $H|_{\text{Fut}}$, it depends on

$$B = i\,\mathrm{sgn}(iA) = Q\begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{pmatrix}Q^T$$

(A non-degenerate) (ask?)

B skew symmetric,

$B^2 = -1$, gives ground state through

$$\sum_j P_{jk}|\psi\rangle = 0 \quad \forall k,$$

$$P_{jk} = \tfrac{1}{2}(\delta_{jk} - iB_{jk})$$

B corresponds to a pairing between Majorana modes,

$\phi' = F(x')$ & $\phi'' = F(x'')$ are paired if $x'' = \pm Bx'$ (real in pref. of Bases).

P can be called the "spectral" operator.

projects $\mathbb{C}^{2m}$ to L, where

$$L = \bigoplus_{\lambda \in \sigma(iA) \cap (-\infty, 0)} \ker(iA - \lambda),$$

$\forall z \in L \quad F(z)|\chi\rangle = 0$, so L is the space of annihilators. $z_j, z_j' \in L \Rightarrow \sum_j z_j z_j' = 0$

- choosing a subspace $L \Longleftrightarrow$ choosing B.

- choosing a subspace $L \Longleftrightarrow$ choosing $B$.

$H$ groundstate also characterized by correlation functions $\langle\Psi| c_j c_k |\Psi\rangle = 2B_{kj}$, higher order by wick.

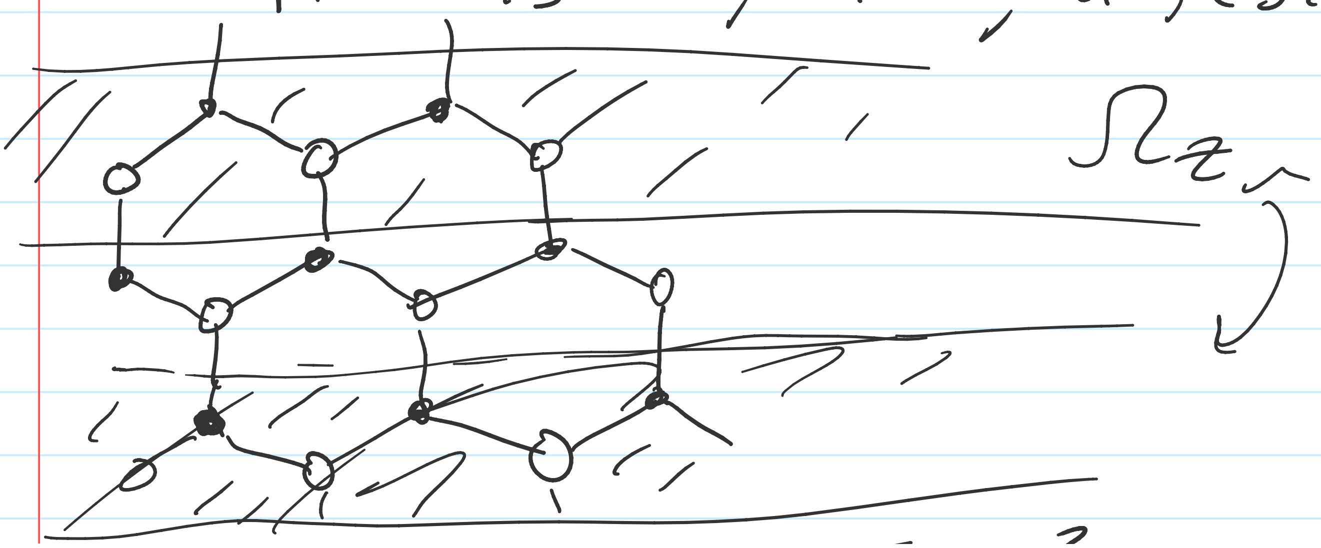
## Fermionic spectrum:

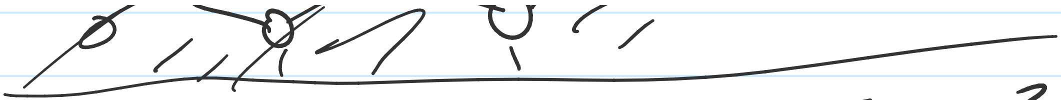
$H$ is parametrized by $u_{jk}$, but it depends on $w_p$.

Global ground state does <u>not</u> depend on $J_x, J_y, J_z$ signs, as sign changes can be accounted for w/ $u_{jk}$ choice, or even fixed $H_u$ ground state.
swap $J_z$ to $-J_z$ corresponds to swapping $u_{jk}$ to $u_{kj}$ $\forall$ $a_{jk} = z$.

But as $w_p$ are preserved, can apply gauge tf to re-obtain $u_{jk}$.

Net effect is $c_j \mapsto -c_j$ at $j \in \Omega_z$:





Obtained by $R_Z = \prod_{i \in \Omega_Z} \sigma_i^z$

Focus on $u$ s.t. $E_u$ minimize

Claim: this occurs when all $w_p = 1$

This follows from Liebs theorem...
pass to Alaric.